M208/SPECIMEN

Module Examination

Pure Mathematics

Time allowed: 3 hours

There are **three sections** in this examination.

In **Section 1** you should **attempt all 10 questions**
This section is worth 70% of the total mark.

In **Section 2** you should **attempt 1 out of the 3 questions**.
Each question is worth 15% of the total mark.

In **Section 3** you should **attempt 1 out of the 2 questions**.
Each question is worth 15% of the total mark.

Include all your working, as some marks are awarded for this.
Write your answers in the answer book(s) provided in **pen**, though you may draw diagrams in pencil.
Start your answer to each question on a new page, clearly indicating the number of the question.
Crossed out work will not be marked.

Calculators are NOT permitted in this examination.

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Section 1

You should **attempt all questions**. **This section is worth 70%.**

Introduction questions (Book A)

Question 1 – 5 marks

This question concerns the complex numbers

$$z_1 = -2 + 2i \quad \text{and} \quad z_2 = 1 + 3i.$$

(a) Draw a diagram showing z_1 and z_2 in the complex plane. [*1*]

(b) Find the modulus and principal argument of z_1. [*2*]

(c) Express $\dfrac{z_1}{z_2}$ in Cartesian form. [*2*]

Question 2 – 5 marks

This question concerns the following subsets of the plane $\mathbb{R}^2$:

$$A = \{(x, y) : x^2 + y^2 \leq 4\}$$
$$B = \{(x, y) : x \leq 2\}.$$

Prove that A is a proper subset of B. [*5*]

Linear algebra questions (Book C)

Question 3 – 6 marks

This question concerns the system of linear equations

$$\begin{aligned} x - y + z &= 1 \\ 2x + 3y - z &= 3 \\ x + 4y - 2z &= 2. \end{aligned}$$

(a) Write down the augmented matrix of this system of linear equations. [*1*]

(b) Determine the row-reduced form of the matrix that you wrote down in part (a). [*3*]

(c) Use your answer to part (b) to solve the system of linear equations. [*2*]

Question 4 – 6 marks

Find the matrix of the linear transformation

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (3x - 4y, 2x + y) \end{aligned}$$

with respect to each of the following.

(a) The standard basis for both the domain and the codomain. [*1*]

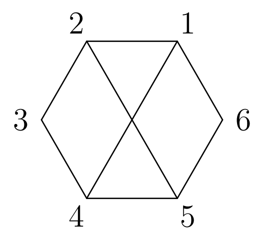
(b) The basis $\{(2, 1), (1, 1)\}$ for the domain and the standard basis for the codomain. [*2*]

(c) The basis $\{(2, 1), (1, 1)\}$ for both the domain and the codomain. [*3*]

Group theory questions (Books B and E)

Question 5 – 8 marks

This question concerns the symmetry group G of the figure below, which is a regular hexagon with two of its diagonals drawn.



(a) Write down all the elements of G in cycle form using the labelling of the vertex locations shown, and describe each element geometrically. [*5*]

(b) Write down a subgroup H of the symmetric group S_6 that is not equal to G but is conjugate to G in S_6, giving its elements in cycle form. State a permutation in S_6 that conjugates G to H. [*3*]

Question 6 – 8 marks

Let

$$G = \{2, 4, 8, 10, 14, 16\}.$$

Then $(G, \times_{18})$ is a group.

(a) Find the identity element of G. [*2*]

(b) Show that $H = \{8, 10\}$ is a subgroup of G. [*2*]

(c) Explain why H is normal in G, and find its cosets in G. [*3*]

(d) State a standard group that is isomorphic to the quotient group G/H. [*1*]

Question 7 – 8 marks

Let

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Then $(G, +)$ is a group.

This question concerns the mapping

$$\begin{aligned} \phi : (G, +) &\longrightarrow (\mathbb{R}, +) \\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &\longmapsto a + b. \end{aligned}$$

(a) Prove that ϕ is a homomorphism. [2]

(b) Determine the kernel and image of ϕ. [4]

(c) State a standard group that is isomorphic to the quotient group $G/\operatorname{Ker}\phi$, justifying your answer briefly. [2]

Analysis questions (Books D and F)

Question 8 – 8 marks

Determine whether each of the following series is convergent, naming any result or test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{4n^2+1}$ [4]

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 e^n}{(n+1)!}$ [4]

Question 9 – 8 marks

(a) Sketch the graph of the function

$$f(x) = \begin{cases} x^2, & x \le 1, \\ 3x - 2, & x > 1. \end{cases}$$ [1]

(b) State whether f is continuous at 1, and prove your statement. [4]

(c) State whether f is differentiable at 1, and prove your statement. [3]

Question 10 – 8 marks

(a) Determine the Taylor polynomials $T_1(x)$ and $T_2(x)$ at 2 for the function

$$f(x) = \frac{x}{x-1}.$$ [5]

(b) Show that $T_2(x)$ approximates $f(x)$ with an error less than or equal to $\dfrac{1}{64}$ on the interval $[2, 2.25]$. [3]

Section 2

You should attempt **one question**. If you attempt more, the score from your best question will count.

Each question is worth 15%.

Question 11

This question concerns the following subset of $\mathbb{R}^3$:

$$S = \{(a+b, -a+2b, 2a-b) : a, b \in \mathbb{R}\}.$$

(a) Show that S is a subspace of $\mathbb{R}^3$. *[5]*

(b) Show that $\{(1,-1,2), (1,2,-1)\}$ is a basis for S, and write down the dimension of S. *[4]*

(c) Find an orthogonal basis for S that contains the vector $(1,-1,2)$. *[3]*

(d) Find the coordinates of $(2,1,1)$ with respect to the basis you found in part (c). *[3]*

Question 12

This question concerns the permutation $p = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$ in S_8.

(a) State the order and the parity of p. *[1]*

(b) (i) Write down the cyclic subgroup $\langle p \rangle$ of S_8 generated by p, giving its elements in cycle form.

(ii) Determine all the subgroups of $\langle p \rangle$, giving the elements of each proper subgroup in cycle form and explaining how you know that you have found all the subgroups. *[8]*

(c) (i) Justify the assertion that $\langle p \rangle$ is a subgroup of the alternating group A_8.

(ii) Write down two elements of S_8 other than p itself that have the same order as p but whose cycle structures differ from each other and from the cycle structure of p.

(iii) Write down a subgroup of S_8 that has the same order as $\langle p \rangle$ but is not a subgroup of A_8, giving its elements in cycle form and justifying your answer. *[6]*

Question 13

(a) Determine whether each of the following sequences (a_n) converges, naming any result or rule that you use. If a sequence does converge, then find its limit.

(i) $a_n = \dfrac{3^n + n!}{n! + 2^n}$

(ii) $a_n = \dfrac{3^n + n + 1}{2n + 2^n - 3}$

(iii) $a_n = \dfrac{(-1)^n n}{2n + 1}$ [*11*]

(b) Determine the least upper bound of the set

$$E = \left\{ 1 - \frac{2}{n^2} : n = 1, 2, \ldots \right\}.$$ [*4*]

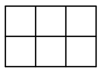
Section 3

You should attempt **one question**. If you attempt more, the score from your better question will count.

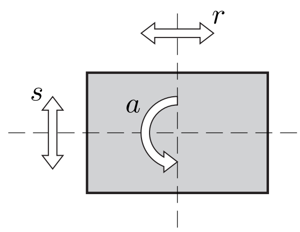
Each question is worth 15%.

Question 14

This question concerns the natural action of the group $S(\square)$ on the set X whose elements are all the figures obtained by colouring each of the six small squares in the figure on the left below either black or white. Some elements of X are shown on the right below.



The group table of $S(\square)$ is given below.



$\circ$	e	a	r	s
e	e	a	r	s
a	a	e	s	r
r	r	s	e	a
s	s	r	a	e

(a) For each of the two elements of X shown below, draw each of the elements in its orbit, and list the elements of its stabiliser.

[5]

(b) (i) Find the size of $\operatorname{Fix} g$ for each symmetry g in $S(\square)$.

(ii) Hence use the Counting Theorem to determine how many different figures there are of the type described above when we regard two of them as the same if one can be obtained by rotating or reflecting the other. [7]

(c) Let $\wedge$ be the group action described above. Suppose that A and B are elements of X such that

$$a \wedge A = B \quad \text{and} \quad s \wedge A = A.$$

Show that $r \wedge A = B$. [3]

Question 15

(a) (i) Use the ε–δ definition of continuity to prove that the function

$$f(x) = x^2 - 3x$$

is continuous at the point $c = 2$.

(ii) Determine whether the function

$$f(x) = \frac{2x}{x^2 - 3}$$

is uniformly continuous on the interval $[-1, 1]$, justifying your answer. [7]

(b) Prove that the following limit exists, and determine its value.

$$\lim_{x \to 0} \frac{\sin(x^2)}{2e^x - e^{2x} - 1}$$ [8]

[END OF QUESTION PAPER]